HOMOGENEOUS SPIRAL MOTION BETWEEN TWO COAXIAL CONES

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The problem of homogeneous spiral motion in a conical vessel of finite dimensions has been studied with application to hydrocyclones [1]. In the operation of the hydrocyclone a column of air forms at the axis. It is therefore of interest to obtain a solution of the problem outside the region of the air column. We consider below the problem of homogeneous spiral motion in a doubly-connected region between two coaxial cones.

1. We shall assume that an ideal incompressible fluid flows between two coaxial cones in a uniform spiral, symmetrical about the common axis. Through the annular crevice around the base of the external cone, fluid enters the cone at the rate of q units per second, whilst through the common vertex of the cones and through the annular crevice encircling the axis it issues at the rates of q_1 and q_2 units per second, respectively. The length of the generator of the cone is equal to R_0, and the complete angles at the vertex (Fig. 1) are equal to $2\theta_1$ and $2\theta_2$, respectively.

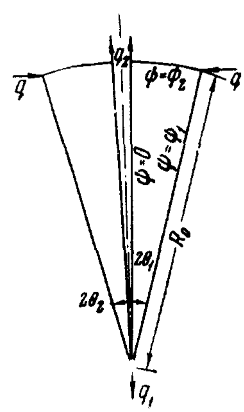

Fig. 1.

The problem reduces to the solution of the inhomogeneous differential equation [2]

$$\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) + k^2\psi = -kC \qquad (k, C = \text{const}) \qquad (1.1)$$

in the region $0 \leqslant r \leqslant R_0$, $\theta_1 \leqslant \theta \leqslant \theta_2$, under the following boundary conditions:

$$\psi(r, \theta_1) = 0, \qquad \psi(r, \theta_2) = \psi_1 = -\frac{q_1}{2\pi} \quad \psi(R_0, \theta) = \psi_2 = \frac{q_2}{2\pi} \qquad (1.2)$$

Instead of the function $\psi(r, \theta)$ let us introduce a new function $u(r,\theta)$ connected to the first by the relation

$$u(r, \theta) = \psi(r, \theta) - \psi_1 \left(\frac{\sin^2\theta - \sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1}\right) \tag{1.3}$$

Equation (1.1) takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial\theta}\right) + k^2 u =$$
$$= -kC + \frac{k^2\psi_1 \sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1} - \frac{\psi_1}{\sin^2\theta_2 - \sin^2\theta_1}\left(k^2 - \frac{2}{r^2}\right)\sin^2\theta \tag{1.4}$$

For the function $u(r, \theta)$ we have the following boundary conditions:

$$u(r, \theta_1) = 0, \qquad u(r, \theta_2) = 0, \qquad u(R_0, \theta) = \psi_2 - \psi_1\left(\frac{\sin^2\theta - \sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1}\right) \tag{1.5}$$

The function $u(r, \theta)$ will be sought in the form of a series

$$u(r, \theta) = M_1(r)L_1(\theta) + M_2(r)L_2(\theta) + \ldots + M_n(r)L_n(\theta) + \ldots \tag{1.6}$$

expanded in the characteristic functions $L_n(\theta)$:

$$L_n(\theta) = \sin\theta\,[Q^1_{\nu_n}(\cos\theta_1)\,P^1_{\nu_n}(\cos\theta) - P^1_{\nu_n}(\cos\theta_1)\,Q^1_{\nu_n}(\cos\theta)]$$

Here $P^1_{\nu_n}(\cos\theta)$, $Q^1_{\nu}(\cos\theta)$ are Legendre's associated functions.

The characteristic numbers ν_n are determined as the roots of the transcendental equation

$$Q^1_\nu(\cos\theta_1)\,P^1_\nu(\cos\theta_2) - P^1_\nu(\cos\theta_1)\,Q^1_\nu(\cos\theta_2) = 0 \tag{1.7}$$

The function $M_n(r)$ is determined from the differential equation [1]

$$\frac{d^2M_n}{dr^2} + \left[k^2 - \frac{\nu_n(\nu_n+1)}{r^2}\right]M_n = -\frac{1}{L_n^2}\left(kC + \psi_1 k^2 b_n - \right.$$
$$\left. - \frac{\psi_1 k^2 \sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1}a_n\right) + \frac{\psi_1}{L_n^2}b_n\frac{2}{r^2} \tag{1.8}$$

Here

$$L_n^2 = \int_{\theta_1}^{\theta_2} \rho(\theta)[L_n(\theta)]^2 d\theta, \qquad a_n = \int_{\theta_1}^{\theta_2} \rho(\theta) L_n(\theta)\,d\theta$$

$$b_n = \frac{1}{\sin^2\theta_2 - \sin^2\theta_1}\int_{\theta_1}^{\theta_2} \sin^2\theta\,\rho(\theta) L_n(\theta)\,d\theta \tag{1.9}$$

The function $\rho(\theta) = \operatorname{cosec}\theta$ represents a weighting function. The function $M_n(r)$ must satisfy the following conditions:

1) as $r \to 0$ the function $M_n(r)$ must be finite in value;

2) when $r = R_0$ according to the third condition (1.5) and the definitions (1.9)

$$M_n(R_0) = \frac{a_n\psi_2}{L_n^2} + \frac{a_n\psi_1}{L_n^2}\,\frac{\sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1} - \frac{b_n\psi_1}{L_n^2} \qquad (1.10)$$

Let us introduce the dimensionless quantities $\rho = r/R_0$, $\kappa = kR_0$ and $\gamma = \psi_2/\psi_1$, and write down the final expression for the stream function

$$\frac{\psi(\rho,\theta)}{\psi_1} = \frac{\sin^2\theta - \sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1} + \sum_{n=1}^{\infty}\left\{\left[a_n\gamma + a_n\frac{\sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1} - b_n + \right.\right.$$
$$+\left(b_n - a_n\frac{\sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1}\right)\sqrt{\varkappa}\, s_{1/2,\mu_n}(\varkappa) - 2b_n\sqrt{\varkappa}\, s_{-3/2,\mu_n}(\varkappa)\Big]\sqrt{\rho}\,\frac{J_{\mu_n}(\varkappa\rho)}{J_{\mu_n}(\varkappa)} -$$
$$\left.-\left(b_n - a_n\frac{\sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1}\right)\sqrt{\varkappa\rho}\, s_{1/2,\mu_n}(\varkappa\rho) + 2b_n\sqrt{\varkappa\rho}\, s_{-3/2,\mu_n}(\varkappa\rho)\right\}\frac{L_n(\theta)}{L_n^2} +$$
$$+\frac{C}{k\psi_1}\sum_{n=1}^{\infty} a_n\left[\sqrt{\varkappa}\, s_{1/2,\mu_n}(\varkappa)\sqrt{\rho}\,\frac{J_{\mu_n}(\varkappa\rho)}{J_{\mu_n}(\varkappa)} - \sqrt{\varkappa\rho}\, s_{1/2,\mu_n}(\varkappa\rho)\right]\frac{L_n(\theta)}{L_n^2} \qquad (1.11)$$

Here $s_{1/2,\mu_n}(\kappa)$ and $s_{-3/2,\mu_n}(\kappa)$ are Lommel functions ($\mu_n = \nu_n + 1/2$).

From (1.11) we can obtain the solution for some special cases. Thus, $\gamma = 0$ corresponds to the case where the annular crevice is absent along the base of the internal cone, whilst $\gamma = 1$ corresponds to the case where the annular crevice is absent along the base of the external cone and the fluid enters through the annular crevice of the small cone.

The calculation is appreciably simplified if we make use of the asymptotic formulas for the associated Legendre functions, derived in [3]

$$P_\nu^m(\cos\theta) = \left(\nu + \frac{1}{2}\right)^m\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_{-m}\left[\left(\nu + \frac{1}{2}\right)\theta\right]$$
$$Q_\nu^m(\cos\theta) = -\frac{\pi}{2}\left(\nu + \frac{1}{2}\right)^m\left(\frac{\theta}{\sin\theta}\right)^{1/2} N_{-m}\left[\left(\nu + \frac{1}{2}\right)\theta\right] \qquad (1.12)$$

which give good results even when $\nu = 10$.

For cases of practical interest $\theta_2 = 15° - 20°$ and the smallest root of Equation (1.7) exceeds 10, so that we can use Formulas (1.12).

Equation (1.7) takes the form

$$J_1(kx)\,N_1(x) - J_1(x)\,N_1(kx) = 0 \qquad \left(\frac{\theta_2}{\theta_1} = k,\quad \left(\nu + \tfrac{1}{2}\right)\theta_1 = x\right) \tag{1.13}$$

If x_n is the nth root of Equation (1.13), then the corresponding root of Equation (1.7) is found from the relation

$$\nu_n + \frac{1}{2} = \frac{x_n}{\theta_1}$$

Calculating L_n^2, a_n, b_n from Formulas (1.9) and using (1.12), we can rewrite Formula (1.11) in the following form, which is more convenient for numerical computations:

$$\frac{\psi(\rho,\theta)}{\psi_1} = \frac{\sin^2\theta - \sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1} + \sum_{n=1}^{\infty}\left\{\left[\alpha_n\gamma + \alpha_n\frac{\sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1} - \beta_n + \right.\right.$$

$$\left. + \left(\beta_n - \alpha_n\frac{\sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1}\right)\sqrt{\varkappa}\,s_{1/2,\mu_n}(\varkappa) - 2\beta_n\sqrt{\varkappa}\,s_{-3/2,\mu_n}(\varkappa)\right]\sqrt{\rho}\,\frac{J_{\mu_n}(\varkappa\rho)}{J_{\mu_n}(\varkappa)} -$$

$$\left. - \left(\beta_n - \alpha_n\frac{\sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1}\right)\sqrt{\varkappa\rho}\,s_{1/2,\mu_n}(\varkappa\rho) + 2\beta_n\sqrt{\varkappa\rho}\,s_{-3/2,\mu_n}(\varkappa\rho)\right\}\times$$

$$\times\frac{\sqrt{\theta\sin\theta}\,Z_1(\mu_n\theta)}{[\mu_n\theta_2 Z_0(\mu_n\theta_2)]^2 - (2/\pi)^2} + \frac{C}{k\psi_1}\sum_{n=1}^{\infty}\alpha_n\times$$

$$\times\left[\sqrt{s_{1/2,\mu_n}(\varkappa)}\,\sqrt{\rho}\,\frac{J_{\mu_n}(\varkappa\rho)}{J_{\mu_n}(\varkappa)} - \sqrt{\varkappa\rho}\,s_{1/2,\mu_n}(\varkappa\rho)\right]\Bigg\}\frac{\sqrt{\theta\sin\theta}\,Z_1(\mu_n\theta)}{[\mu_n\theta_2 Z_0(\mu_n\theta_2)]^2 - (2/\pi)^2} \tag{1.14}$$

Here

$$Z_1(\mu_n\theta) = J_1(\mu_n\theta_1)\,N_1(\mu_n\theta) - N_1(\mu_n\theta_1)\,J_1(\mu_n\theta)$$

$$Z_0(\mu_n\theta_2) = J_1(\mu_n\theta_1)\,N_0(\mu_n\theta_2) - N_1(\mu_n\theta_1)\,J_0(\mu_n\theta_2)$$

$$\alpha_n = \frac{4}{\pi\sqrt{\theta_1\sin\theta_1}} - 2\mu_n\left(\frac{\theta_2}{\sin\theta_2}\right)^{1/2} Z_0(\mu_n\theta_2)$$

$$\beta_n = \frac{2}{\sin^2\theta_2 - \sin^2\theta_1}\;\frac{\mu_n(\mu_n - 1)(\mu_n + 1/2)}{\mu_n^2 - 9/4}\times$$

$$\times\{\sqrt{\theta_2\sin\theta_2}\,[J_1(\mu_n\theta_1)\,N_1((\mu_n - 1)\,\theta_2) - N_1(\mu_n\theta_1)\,J_1((\mu_n - 1)\,\theta_2)] -$$

$$- \sqrt{\theta_1\sin\theta_1}\,[J_1(\mu_n\theta_1)\,N_1((\mu_n - 1)\,\theta_1) - N_1(\mu_n\theta_1)\,J_1((\mu_n - 1)\,\theta_1)]\}$$

With the help of Formula (1.14) we carried out the computation for the following data:

$$\theta_1 = 1.5^\circ,\qquad \theta_2 = 15^\circ,\qquad \varkappa = 4,\qquad C / k\psi_1 = -15,\ \gamma = -2$$

To estimate the influence of the air core we carried out similar calculations for the complete cone with $\theta_2 = 15°$ and the same values of the parameters κ, $C/k\psi_1, \gamma$.

The results of the calculations show that the presence of the air core has a very insignificant effect on the distribution of the streamlines. Accordingly, in the determination of the flow chart of the fluid motion inside the hydrocyclone, the influence of the air core can be neglected.

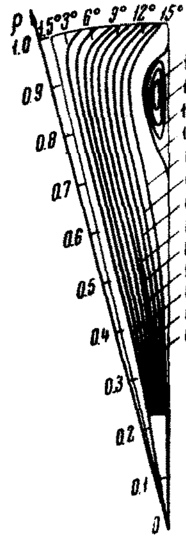

Fig. 2.

2. Now suppose that the rigid spherical surface is absent, whilst ψ_2 is a given function of θ. Instead of the third condition (1.3) we shall have $\psi(R_0, \theta) = f(\theta)$. The function $f(\theta)$ must be such that when $\theta = \theta_1$ it vanishes, whilst it is equal to ψ_1 when $\theta = \theta_2$.

In what follows only the quantity $M_n(R_0)$ is changed:

$$M_n(R_0) = \frac{d_n}{L_n^2} + \frac{a_n}{L_n^2} \frac{\sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1}\psi_1 - \frac{b_n}{L_n^2}\psi_1$$

$$\left(d_n = \int_{\theta_1}^{\theta_1} f(\theta)\rho(\theta)\ L_n(\theta)\,d\theta \right) \qquad (2.1)$$

Let us solve the problem in the particular case when the radial component of velocity is constant at $r = R_0$. It is not difficult to show that in this case

$$f(\theta) = \psi_1 \quad \frac{\cos\theta_1 - \cos\theta}{\cos\theta_1 - \cos\theta_2} \qquad (2.2)$$

We shall write down the final expression for the stream function:

$$\frac{\psi(\rho,\theta)}{\psi_1} = \frac{\sin^2\theta - \sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1} + \sum_{n=1}^{\infty}\left\{\left[\gamma_n - \beta_n + \left(\beta_n - \alpha_n \frac{\sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1}\right)\times\right.\right.$$

$$\left.\times\sqrt{\varkappa}s_{1/2,\mu_n}(\varkappa) - 2\beta_n\sqrt{\varkappa}s_{-3/2,\mu_n}(\varkappa)\right]\sqrt{\rho}\,\frac{J_{\mu_n}(\varkappa\rho)}{J_{\mu_n}(\varkappa)} - \left(\beta_n - \alpha_n \frac{\sin^2\theta_1}{\sin^2\theta_2 - \sin^2\theta_1}\right)\times$$

$$\left.\times\sqrt{\varkappa\rho}\,s_{1/2,\mu_n}(\varkappa\rho) + 2\beta_n\sqrt{\varkappa\rho}\,s_{-3/2,\mu_n}(\varkappa\rho)\right\}\frac{\sqrt{\theta\sin\theta}\,Z_1(\mu_n\theta)}{[\mu_n\theta_2 Z_0(\mu_n\theta_2)]^2 - (2/\pi)^2} +$$

$$+\frac{C}{k\psi_1}\sum_{n=1}^{\infty}\alpha_n\left[\sqrt{\varkappa}\,s_{1/2,\mu_n}(\varkappa)\sqrt{\rho}\,\frac{J_{\mu_n}(\varkappa\rho)}{J_{\mu_n}(\varkappa)} - \sqrt{\varkappa\rho}\,s_{1/2,\mu_n}(\varkappa\rho)\right]\frac{\sqrt{\theta\sin\theta}\,Z_1(\mu_n\theta)}{[\mu_n\theta_2 Z_0(\mu_n\theta_2)]^2 - (2/\pi)^2} \qquad (2.3)$$

where α_n and β_n are as before, and

$$\times \left[\sqrt{s_{1/2,\,\mu_n}}(\varkappa)\,\sqrt{\rho}\,\frac{J_{\mu_n}(\varkappa\rho)}{J_{\mu_n}(\varkappa)} - \sqrt{\varkappa\rho s_{1/2,\,\mu_n}}(\varkappa\rho)\right]\Bigg\}\frac{\sqrt{\theta\sin\theta}\,Z_1(\mu_n\theta)}{[\mu_n\theta_2 Z_0(\mu_n\theta_2)]^2-(2/\pi)^2} \qquad (1.14)$$

From Formula (2.3) we performed the calculations for $\theta_1 = 1.5°$, $\theta_2 = 15°$, $\kappa = 4$ and for various values of the parameter $C/k\psi_1$. The streamlines in the case $C/k\psi_1 = 7$ are shown in Fig. 2.

BIBLIOGRAPHY

1. Bostandzhiian, S.A., Odnorodnoe vintovoe dvizhenie v konuse (Homogeneous spiral motion in a cone). *PMM* Vol. 25, No. 1, 1961.

2. Vasil'ev, O.F., *Osnovy mekhaniki vintovykh i tsirkuliatsionnykh potokov (Fundamental Mechanics of Spiral and Circulatory Flows)*. Gosenergoizdat, 1958.

3. Muradian, R.M., Asimptoticheskie formuly dlia obobshchennykh funktsii Lezhandra i Gegenbauera (Asymptotic formulas for the generalized functions of Legendre and Gegenbauer). *Dokl. Akad. Nauk SSSR* Vol. 115, No. 5, 1957.

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